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Optimal Low-Earth-Orbit–Geostationary-Earth-Orbit Intermediate Acceleration Orbit Transfer

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The problem of minimum-time orbit transfer using intermediate acceleration is analyzed using both precision integration and averaging. Continuous constant accelerations of the order of 10^{-2} g are considered for applications using nuclear propulsion upper stages. The acceleration vector is optimized in direction with its magnitude held constant throughout the flight. These trajectories that circle the Earth for only a few orbits before reaching geostationary Earth orbit are shown to be sensitive to departure and arrival points, necessitating the use of the full six-state dynamics for satisfactory and meaningful results. The ΔV losses with respect to very low-acceleration transfers are shown to be small.

I. Introduction

THE problem of optimal low-thrust orbit transfer has generated hundreds of technical contributions from many innovative applied mathematicians and astrodynamicists in the past four decades. A representative set of this abundant literature using orbit-element-based formulations and the classical methods of the calculus of variations is shown in Refs. 1–16. The need for fast and efficient computer codes useful in mission analysis led Edelbaum[7] and Edelbaum et al.[10] to apply the technique of averaging to solve the general ellipse-to-ellipse transfer problem with continuous thrust. In particular, the use of orbit parameters in the form of the nonsingular equinoctial elements developed by Broucke and Cefola[8] has allowed Edelbaum et al.[10] to create the computer codes to solve the five-state transfer problem. The full six-state formulation appeared in Refs. 17–19 to solve the minimum-time rendezvous using continuous constant acceleration. However, when intermediate acceleration levels are considered, the five-state transfer becomes dependent on the initial and final locations along the corresponding orbits, thereby necessitating the use of the six-state theory.

This paper presents results of precision-integrated, optimized low-Earth-orbit (LEO)-to-geostationary-Earth-orbit (GEO) minimum-time transfer and compares them to the solutions obtained by way of the averaging technique. A 10^{-2}-g acceleration applied in a constant and continuous manner is taken as an example to generate fast subday transfers that could be flown with nuclear thermal propulsion upper stages. The calculus of variations formulation adopted here makes use of a set of nonsingular orbit elements with the current mean longitude as the sixth state variable. This six-state formulation allows us to generate optimal transfers that first start from a given fixed location on the initial orbit while optimizing the arrival point on the target or final orbit. The analysis is extended further to optimize both departure and arrival points to obtain the overall minimum-time free-free solution. This requires the vanishing of the Lagrange multiplier adjoint to the mean longitude at both initial and final times with fixed initial time and optimized final time. These fast, few-revolution, five-state transfers are shown to be sensitive to initial and final orbital position, thereby necessitating the use of the full six-state dynamics. These exact results then are compared to the approximate solutions obtained using averaged dynamics with robust and fast convergence characteristics. It is shown that the ΔV or transfer time solutions compare rather well, even for these short duration transfers but that the element time histories, and especially the eccentricity, are poorly simulated by the approximate solutions. Furthermore, due to the nature of the averaging technique, the sensitivity of the solution to orbital position is totally removed such that the precision-integrated solution must be used instead for accurate guidance.

II. Minimum-Time Transfer from Fixed Initial State with Continuous Constant Acceleration

If we assume a continuous constant acceleration $f_t = f/m$, where f represents thrust and m the vehicle mass, and use the set of equinoctial orbit elements a, h, k, p, q, and λ, which are related to the classical elements by $h = e\sin(\omega+\Omega)$, $k = e\cos(\omega+\Omega)$, $p = \tan(i/2)\sin\Omega$, $q = \tan(i/2)\cos\Omega$, and $\lambda = M+\omega+\Omega$, the equations of motion of a thrusting vehicle are given by

$$\dot{a} = \left(\frac{\partial a}{\partial \dot{\boldsymbol{r}}}\right)^T \cdot \hat{\boldsymbol{u}} f_t \tag{1}$$

$$\dot{h} = \left(\frac{\partial h}{\partial \dot{\boldsymbol{r}}}\right)^T \cdot \hat{\boldsymbol{u}} f_t \tag{2}$$

$$\dot{k} = \left(\frac{\partial k}{\partial \dot{\boldsymbol{r}}}\right)^T \cdot \hat{\boldsymbol{u}} f_t \tag{3}$$

$$\dot{p} = \left(\frac{\partial p}{\partial \dot{\boldsymbol{r}}}\right)^T \cdot \hat{\boldsymbol{u}} f_t \tag{4}$$

$$\dot{q} = \left(\frac{\partial q}{\partial \dot{\boldsymbol{r}}}\right)^T \cdot \hat{\boldsymbol{u}} f_t \tag{5}$$

$$\dot{\lambda} = n + \left(\frac{\partial \lambda}{\partial \dot{\boldsymbol{r}}}\right)^T \cdot \hat{\boldsymbol{u}} f_t \tag{6}$$

where $\hat{\boldsymbol{u}}$ is a unit vector in the direction of the thrust, n is the orbit mean motion, and

$$\frac{\partial a}{\partial \dot{\boldsymbol{r}}} = 2a^{-1}n^{-2}(\dot{X}_1\hat{f} + \dot{Y}_1\hat{g}) = M_{11}\hat{f} + M_{12}\hat{g} + M_{13}\hat{w} \tag{7}$$

$$\frac{\partial h}{\partial \dot{\boldsymbol{r}}} = Gn^{-1}a^{-2}\left[\left(\frac{\partial X_1}{\partial k} - h\beta\frac{\dot{X}_1}{n}\right)\hat{f} + \left(\frac{\partial Y_1}{\partial k} - h\beta\frac{\dot{Y}_1}{n}\right)\hat{g}\right] + k(qY_1 - pX_1)n^{-1}a^{-2}G^{-1}\hat{w} = M_{21}\hat{f} + M_{22}\hat{g} + M_{23}\hat{w} \tag{8}$$

$$\frac{\partial k}{\partial \dot{\boldsymbol{r}}} = -Gn^{-1}a^{-2}\left[\left(\frac{\partial X_1}{\partial h} + k\beta\frac{\dot{X}_1}{n}\right)\hat{f} + \left(\frac{\partial Y_1}{\partial h} + k\beta\frac{\dot{Y}_1}{n}\right)\hat{g}\right] - h(qY_1 - pX_1)n^{-1}a^{-2}G^{-1}\hat{w} = M_{31}\hat{f} + M_{32}\hat{g} + M_{33}\hat{w} \tag{9}$$

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$$\frac{\partial p}{\partial \dot{\boldsymbol{r}}} = KY_1 \frac{n^{-1}a^{-2}G^{-1}}{2}\hat{w} = M_{41}\hat{f} + M_{42}\hat{g} + M_{43}\hat{w} \tag{10}$$

$$\frac{\partial q}{\partial \dot{\boldsymbol{r}}} = KX_1 \frac{n^{-1}a^{-2}G^{-1}}{2}\hat{w} = M_{51}\hat{f} + M_{52}\hat{g} + M_{53}\hat{w} \tag{11}$$

$$\frac{\partial \lambda}{\partial \dot{\boldsymbol{r}}} = n^{-1}a^{-2}\left[-2X_1 + G\left(h\beta\frac{\partial X_1}{\partial h} + k\beta\frac{\partial X_1}{\partial k}\right)\right]\hat{f} + n^{-1}a^{-2}\left[-2Y_1 + G\left(h\beta\frac{\partial Y_1}{\partial h} + k\beta\frac{\partial Y_1}{\partial k}\right)\right]\hat{g} + n^{-1}a^{-2}G^{-1}(qY_1 - pX_1)\hat{w} = M_{61}\hat{f} + M_{62}\hat{g} + M_{63}\hat{w} \tag{12}$$

The $(\hat{f}, \hat{g}, \hat{w})$ frame is the so-called direct equinoctial frame in which both $\hat{\boldsymbol{u}}$ and the velocity partials are expressed. The 6×3 M matrix is dependent on the elements, as well as the eccentric longitude F. The position and velocity vectors are given in terms of the eccentric longitude F, which is itself related to the eccentric anomaly E by $F = E + \tan^{-1}(h/k)$,

$$\boldsymbol{r} = X_1\hat{f} + Y_1\hat{g} \qquad \dot{\boldsymbol{r}} = \dot{X}_1\hat{f} + \dot{Y}_1\hat{g}$$

with

$$X_1 = a\left[(1 - h^2\beta)c_F + hk\beta s_F - k\right]$$

$$Y_1 = a\left[hk\beta c_F + (1 - k^2\beta)s_F - h\right]$$

$$\dot{X}_1 = na^2r^{-1}\left[hk\beta c_F - (1 - h^2\beta)s_F\right]$$

$$\dot{Y}_1 = na^2r^{-1}\left[(1 - k^2\beta)c_F - hk\beta s_F\right]$$

and where $\beta = [1/(1+G)]$, $G = (1 - h^2 - k^2)^{1/2}$, $K = (1 + p^2 + q^2)$, and $r = a(1 - kc_F - hs_F)$. As λ is being integrated, it is necessary to solve for F from Kepler's transcendental equation by iteration, namely, from $\lambda = F - ks_F + hc_f$. Finally,

$$\frac{\partial X_1}{\partial h} = a\left[-(hc_F - ks_F)\left(\beta + \frac{h^2\beta^3}{1-\beta}\right) - \frac{a}{r}c_F(h\beta - s_F)\right]$$

$$\frac{\partial X_1}{\partial k} = -a\left[(hc_F - ks_F)\frac{hk\beta^3}{1-\beta} + 1 + \frac{a}{r}s_F(s_F - h\beta)\right]$$

$$\frac{\partial Y_1}{\partial h} = a\left[(hc_F - ks_F)\frac{hk\beta^3}{1-\beta} - 1 + \frac{a}{r}c_F(k\beta - c_F)\right]$$

$$\frac{\partial Y_1}{\partial k} = a\left[(hc_F - ks_F)\left(\beta + \frac{k^2\beta^3}{1-\beta}\right) + \frac{a}{r}s_F(c_F - k\beta)\right]$$

The Hamiltonian of this differential system is written as

$$H = \boldsymbol{\lambda}_z^T M(z, F) f_t \hat{\boldsymbol{u}} + \lambda_\lambda n = \boldsymbol{\lambda}_z^T \dot{z} \tag{13}$$

where $\boldsymbol{\lambda}_z^T = (\lambda_a \;\; \lambda_h \;\; \lambda_k \;\; \lambda_p \;\; \lambda_q \;\; \lambda_\lambda)$ is the vector of Lagrange multipliers adjoint to the state variable $z = (a \;\; h \;\; k \;\; p \;\; q \;\; \lambda)^T$ and the Euler–Lagrange equations are given by

$$\dot{\boldsymbol{\lambda}}_z = -\frac{\partial H}{\partial z} = -\boldsymbol{\lambda}_z^T \frac{\partial M}{\partial z} f_t \hat{\boldsymbol{u}} - \lambda_\lambda \frac{\partial n}{\partial z} \tag{14}$$

The $\partial M / \partial z$ partials are found in the Appendices of Refs. 17 and 19. They are also shown in the Appendix of this paper for completeness. Because we are considering continuous constant acceleration, we need only to optimize the acceleration direction, and this is done by selecting $\hat{\boldsymbol{u}}$ parallel to $\boldsymbol{\lambda}_z^T M(z, F)$ at all times. We minimize total transfer time by maximizing the performance index

$$J = -\int_{t_0}^{t_f} \mathrm{d}t = -(t_f - t_0)$$

For fixed t_0, the minimization of t_f or maximization of $-t_f$ gives rise to the transversality condition $H_f = 1$ for $H = \boldsymbol{\lambda}_z^T \dot{z}$ adopted here. We now use the full six-element formulation to solve five-state orbit transfer problems by starting from a fixed initial state and optimizing the final arrival point on the terminal orbit. Given $(a)_0$, $(e)_0$, $(i)_0$, $(\Omega)_0$, $(\omega)_0$, and $(M)_0$ at time $t_0 = 0$ or, equivalently, $(a)_0$, $(h)_0$, $(k)_0$, $(p)_0$, $(q)_0$, and $(\lambda)_0$, the initial values of the Lagrange multipliers, namely, $(\lambda_a)_0$, $(\lambda_h)_0$, $(\lambda_k)_0$, $(\lambda_p)_0$, $(\lambda_q)_0$, and $(\lambda_\lambda)_0$, are guessed and the dynamic equations $\dot{z} = (\partial z / \partial \dot{\boldsymbol{r}}) \cdot \hat{\boldsymbol{u}} f_t$, as well as the adjoint Eqs. (14), are integrated forward to the guessed transfer time t_f by using the optimal control $\hat{\boldsymbol{u}} = \boldsymbol{\lambda}_z^T M(z, F) / |\boldsymbol{\lambda}_z^T M(z, F)|$. The integrations are carried out with the seventh-order variable-step Runge–Kutta–Fehlberg RK78 integrator with the relative and absolute error controls set at 10^{-9}. An iterative scheme based on a general descent method is used to adjust the initial values of the six multipliers, as well as t_f, such that the five terminal state parameters a_f, h_f, k_f, p_f, and q_f are matched and $(\lambda_\lambda)_f = 0$, $H_f = 1$ are satisfied. This is done by minimizing the following objective function:

$$F' = w_1(a - a_f)^2 + w_2(h - h_f)^2 + w_3(k - k_f)^2 + w_4(p - p_f)^2 + w_5(q - q_f)^2 + w_6\left(\lambda_\lambda - \lambda_{\lambda_f}\right)^2 + w_7(H - H_f)^2 \tag{15}$$

or

$$F' = \sum_{i=1}^{5} w_i\left(z_i - z_{i_f}\right)^2 + w_6(\lambda_\lambda - 0)^2 + w_7(H - 1)^2 \tag{16}$$

with w_i being certain weights that can be adjusted to favor the rapid convergence of some elements relative to others and alleviate, to some extent, certain sensitivity and scaling problems associated with the use of a given optimizer. Let the initial orbit be given by $a_0 = 7000$ km, $e_0 = 0$, $i_0 = 28.5$ deg, $\Omega_0 = 0$ deg, $\omega_0 = 0$ deg, and $M_0 = -220$ deg, and let $f_t = 9.8 \times 10^{-5}$ km/s^2 or roughly 10^{-2} g. The final orbit is given by $a_f = 42{,}000$ km, $e_f = 10^{-3}$, $i_f = 1$ deg, $\Omega_f = 0$ deg, and $\omega_f = 0$ deg with M_f free. The following solution is obtained: $(\lambda_a)_0 = 1.260484756$ s/km, $(\lambda_h)_0 = 3.865626962 \times 10^2$ s, $(\lambda_k)_0 = -9.388262635 \times 10^3$ s, $(\lambda_p)_0 = -2.277132367 \times 10^3$ s, $(\lambda_q)_0 = -1.743027218 \times 10^4$ s, $(\lambda_\lambda)_0 = 5.155487187 \times 10^2$ s/rad, and $t_f = 58{,}624.094$ s. The achieved orbit parameters at $t = t_f$ are $a_f = 42{,}000.007$ km, $e_f = 1.00022 \times 10^{-3}$, $i_f = 1.000012$ deg, $\Omega_f = 359.999963$ deg, $\omega_f = 1.966524$ deg $\times$ 10^{-2}, and $M_f = 43.779715$ deg, with $H_f = 1.002694$ and $(\lambda_\lambda)_f = -7.662506 \times 10^{-3}$ s/rad. The value of $\lambda_f = 19.613998$ rad is such that with $\lambda_0 = -3.839724$ rad, a total travel of $\Delta\lambda/(2\pi) = (\lambda_f - \lambda_0)/(2\pi) = 23.453722/(2\pi) = 3.732$ revolutions around the Earth is accomplished. The initial values of the equinoctial elements are given by $a_0 = 7000$ km, $h_0 = 0$, $k_0 = 0$, $p_0 = 0$, $q_0 = 0.2539676465$, and $\lambda_0 = -3.839724354$ rad. Figure 1 shows the variations of the classical elements a and e as a function of time during the 16.284470-h transfer. The eccentricity reaches a peak of around 0.4 with most of the orbit rotation taking place in the final 2 h of the transfer. The final position given by λ_f or M_f with $M_f = 43.779715$ deg is the optimal arrival point on the target orbit that results in the minimum-time transfer for the given or fixed initial $M_0 = -220$ deg.

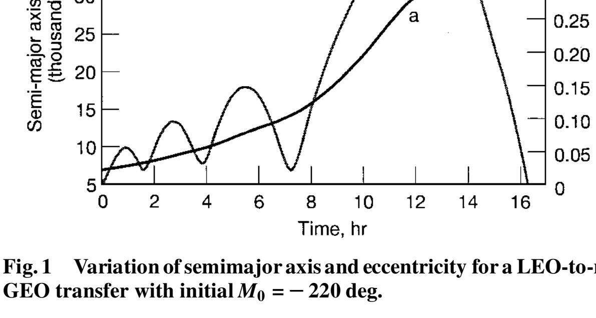

Fig. 1 Variation of semimajor axis and eccentricity for a LEO-to-near-GEO transfer with initial $M_0 = -220$ deg.

The total $\Delta V = f_t t_f = 5.745161$ km/s. We now solve the same transfer problem except that $M_0 = -300$ deg instead of -220 deg on the initial orbit at LEO. This will show whether there exists any sensitivity related to the departure location on the initial orbit. The solution now is given by $(\lambda_a)_0 = 7.131919334$ s/km, $(\lambda_h)_0 = -6.586531878 \times 10^2$ s, $(\lambda_k)_0 = 9.135791613 \times 10^3$ s, $(\lambda_p)_0 = 1.439828925 \times 10^3$ s, $(\lambda_q)_0 = -2.414079986 \times 10^4$ s, $(\lambda_\lambda)_0 = -3.736240460 \times 10^2$ s/rad, and $t_f = 58158.832$ s. This transfer requires a flight time of some 465.262 s less than the previous transfer. The achieved parameters are $a_f = 42{,}000.012$ km, $e_f = 9.9965 \times 10^{-4}$, $i_f = 1.000030$ deg, $\Omega_f = 359.998924$ deg, $\omega_f = 359.975378$ deg, and $M_f = 227.155823$ deg, with $H_f = 1.000000000$ and $(\lambda_\lambda)_f = -8.872380 \times 10^{-3}$ s/rad and $\lambda_0 = -5.235987$ rad and $\lambda_f = 16.530539$ rad such that the total angular travel consists of $(\lambda_f - \lambda_0)/(2\pi) = 3.464$ revolutions around the Earth. The variations of the classical orbit elements a and e during the optimal transfer are similar to the ones corresponding to the first example depicted in Fig. 1, with smaller oscillation amplitudes in the eccentricity during the first revolutions. The total $\Delta V = f_t \cdot t_f = 5.699565$ km/s.

III. Minimum-Time Transfer with Optimized Departure and Arrival Locations

In this problem, we are given a_0, e_0, i_0, Ω_0, and ω_0 or, equivalently, $(a)_0$, $(h)_0$, $(k)_0$, $(p)_0$, and $(q)_0$. The guessed parameters are $(\lambda_a)_0$, $(\lambda_h)_0$, $(\lambda_k)_0$, $(\lambda_p)_0$, $(\lambda_q)_0$, and the initial mean longitude $(\lambda)_0$, as well as the transfer time t_f. The precision integration of the dynamic and adjoint systems in Eqs. (1–6) and (14), respectively, is carried out using $(\lambda_\lambda)_0 = 0$. The boundary conditions at the unknown final time are given by $a_f, h_f, k_f, p_f, q_f, (\lambda_\lambda)_f = 0$, and $H_f = 1$ for a minimum-time solution. The optimal thrust direction still is given by $\hat{\boldsymbol{u}} = \boldsymbol{\lambda}_z^T M(z, F)/|\boldsymbol{\lambda}_z^T M(z, F)|$. The objective function to minimize still is given in Eq. (16). We have essentially replaced the boundary condition (λ_0) of the preceding section by the new boundary condition $(\lambda_\lambda)_0 = 0$, inasmuch as small variations in $(\lambda)_0$ must be such that the performance index $J = -t_f$ is stationary indicating zero sensitivity to initial location. The same argument still holds true for the arrival point on the target orbit. The solution of the optimal free-free transfer is given by $(\lambda_a)_0 = 4.8548563957$ s/km, $(\lambda_h)_0 = 5.52370740318 \times 10^2$ s, $(\lambda_k)_0 = -9.51431194293 \times 10^3$ s, $(\lambda_p)_0 = -1.0373235843 \times 10^2$ s, $(\lambda_q)_0 = -2.33561012603 \times 10^4$ s, $(\lambda)_0 = -2.272581909$ rad corresponding to $M_0 = -130.209352$ deg, and the overall minimum time $t_f = 58{,}090.031$ s requiring a ΔV of 5.692823 km/s. The total angular travel is obtained from $(\lambda)_0$ and $(\lambda)_f = 19.65399449$ rad or 3.4897 revolutions around the Earth. The final achieved parameters are $a_f = 42{,}000.001$ km, $e_f = 9.78045 \times 10^{-4}$, $i_f = 0.999359$ deg, $\Omega_f = 358.777222$ deg, $\omega_f = 350.922884$ deg, and $M_f = 56.390827$ deg with $H_f = 1.038422077$ and $(\lambda_\lambda)_f = -8.6830309 \times 10^{-1}$ s/rad. The Hamiltonian is constant because the dynamic equations are not explicit functions of time. Furthermore, we can multiply all of the initial values of the multipliers by a common constant number to get $H = 1$ exactly, if we so desire. The solution is, of course, unchanged because of the common scaling used. This also means that we can arbitrarily select one of the λ and reduce the order of the integration by one because H, a positive constant, is in effect a first integral of the motion. The initial values of the multipliers that scale the Hamiltonian to the unit value are $(\lambda_a)_0 = 4.675224557$ s/km, $(\lambda_h)_0 = 5.319327780 \times 10^2$ s, $(\lambda_k)_0 = -9.162278183 \times 10^3$ s, $(\lambda_p)_0 = -9.989421518 \times 10^1$ s, and $(\lambda_q)_0 = -2.249191516 \times 10^4$ s. Figure 2 depicts the evolution of e and i as a function of the semimajor axis. Most of the inclination change is taking place at or near maximum eccentricity. The time histories of the six Lagrange multipliers are shown in Figs. 3–5. Finally, the thrust pitch and yaw programs are shown in Fig. 6. The yaw profile, as expected, changes sign every one-half revolution during the first three orbits, rotating the orbit slowly, but the pitch angle stays near zero for maximum energy buildup. The sharp buildup in the yaw angle is responsible for most of the orbit rotation, which takes place as soon as eccentricity reaches its maximum value, as already observed. As soon as the orbit reaches the proper energy level, it is able to transfer directly to the GEO

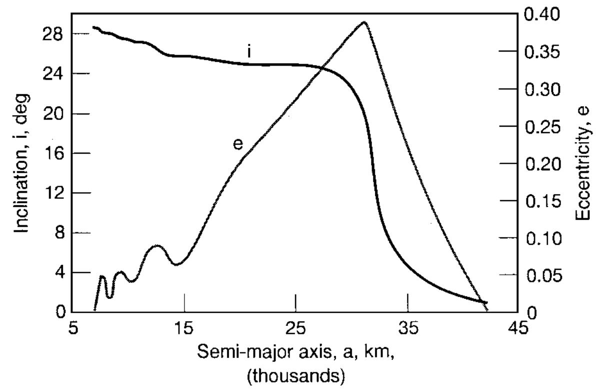

Fig. 2 Eccentricity and inclination vs semimajor axis for absolute minimum-time LEO-to-near-GEO transfer.

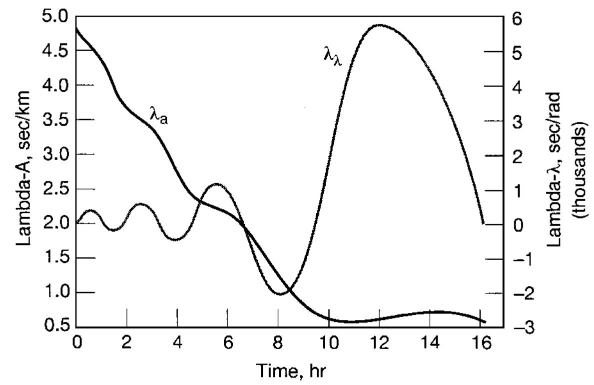

Fig. 3 Time history of λ_a and λ_λ multipliers for absolute minimum-time LEO-to-near-GEO transfer.

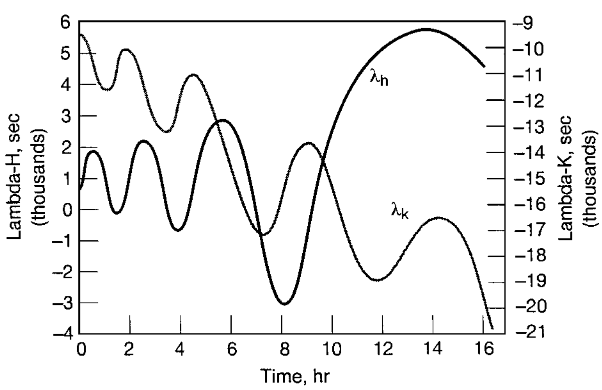

Fig. 4 Time history of λ_h and λ_k multipliers for absolute minimum-time LEO-to-near-GEO transfer.

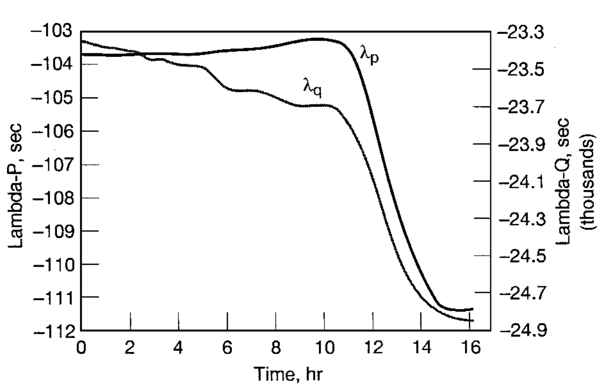

Fig. 5 Time history of λ_p and λ_q multipliers for absolute minimum-time LEO-to-near-GEO transfer.

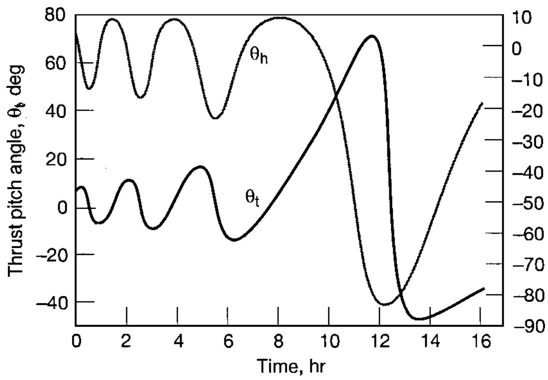

Fig. 6 Optimal thrust pitch and yaw programs for absolute minimum-time LEO-to-near-GEO transfer.

altitude on a highly eccentric orbit. The last portion of the trajectory consists of a circularization maneuver to cancel the eccentricity and enter the GEO orbit. The pitch angle θ_t becomes negative during the last portion of this phase, as shown in Fig. 6 for final eccentricity control. The θ_t is such that $-90 < \theta_t < 90$ deg at all times, indicating that the thrust is never applied in a braking mode. This is not always the case, especially for larger inclination changes or larger orbit transfers.

IV. Continuous Constant Acceleration Orbit Transfer Using Averaging

Following Ref. 10, an averaged Hamiltonian $\tilde{H}$ is formed first from which a first-order approximation to the state and costate is derived by holding these quantities constant over the averaging interval and varying only F, the eccentric longitude, on a conic orbit:

$$\tilde{H} = \frac{1}{T_0}\int_0^{T_0} H\,\mathrm{d}t = \frac{1}{T_0}\int_{-\pi}^{\pi} \frac{H\,\mathrm{d}F}{\dot{F}(\tilde{z},F)} \tag{17}$$

where T_0, the orbital period at time t, is given by $T_0 = 2\pi/\tilde{n}$ with $\tilde{n} = \mu^{1/2}\tilde{a}^{-3/2}$ and where $\tilde{a}$ indicates the averaged value of the semimajor axis at time t, etc. From Kepler's equation $\tilde{\lambda} = F - \tilde{k}s_F + \tilde{h}c_F = \tilde{n}t$, we have

$$\dot{F} = \frac{2\pi}{T_0(1 - \tilde{k}c_F - \tilde{h}s_F)} \tag{18}$$

such that, with $s = (1/T_0)/\dot{F} = [1/(2\pi)](1 - \tilde{k}c_F - \tilde{h}s_F)$, the equations for the approximation for the state and costate variables are given by[10]

$$\dot{\tilde{z}} = \left(\frac{\partial \tilde{H}}{\partial \tilde{\boldsymbol{\lambda}}_z}\right)^T = \int_{-\pi}^{\pi}\left(\frac{\partial H}{\partial \tilde{\boldsymbol{\lambda}}_z}\right)^T s(\tilde{z},F)\,\mathrm{d}F \tag{19}$$

$$\dot{\tilde{\boldsymbol{\lambda}}}_z = -\left(\frac{\partial \tilde{H}}{\partial \tilde{z}}\right)^T = -\int_{-\pi}^{\pi}\left[\left(\frac{\partial H}{\partial \tilde{z}}\right)^T s(\tilde{z},F) + H\left(\frac{\partial s}{\partial \tilde{z}}\right)^T\right]\mathrm{d}F \tag{20}$$

The partials $\partial s/\partial \tilde{z}$ are given by

$$\frac{\partial s}{\partial \tilde{a}} = 0, \qquad \frac{\partial s}{\partial \tilde{h}} = \frac{-s_F}{2\pi}, \qquad \frac{\partial s}{\partial \tilde{k}} = \frac{-c_F}{2\pi}$$

$$\frac{\partial s}{\partial \tilde{p}} = 0, \qquad \frac{\partial s}{\partial \tilde{q}} = 0, \qquad \frac{\partial s}{\partial \tilde{\lambda}} = \frac{\tilde{k}s_F - \tilde{h}c_F}{(1 - \tilde{k}c_F - \tilde{h}s_F)2\pi}$$

because

$$\frac{\partial s}{\partial \tilde{\lambda}} = \frac{\partial s}{\partial F}\cdot\frac{\partial F}{\partial \tilde{\lambda}}, \qquad \tilde{\lambda} = F - \tilde{k}s_F + \tilde{h}c_F$$

The optimal trajectory is such that H is constant throughout. This means that when thrust is applied continuously during every complete revolution, the contribution from the $H(\partial s/\partial \tilde{z})^T$ term is equal to zero because the $\partial s/\partial \tilde{z}$ terms will give a zero net contribution. For a continuous constant acceleration f_t, the averaged equations are given by

$$\dot{\tilde{a}} = \frac{1}{2\pi}f_t\int_{-\pi}^{\pi}\left(\frac{\partial a}{\partial \dot{\boldsymbol{r}}}\right)^T\cdot\hat{\boldsymbol{u}}(1 - \tilde{k}c_F - \tilde{h}s_F)\,\mathrm{d}F \tag{21}$$

$$\dot{\tilde{h}} = \frac{1}{2\pi}f_t\int_{-\pi}^{\pi}\left(\frac{\partial h}{\partial \dot{\boldsymbol{r}}}\right)^T\cdot\hat{\boldsymbol{u}}(1 - \tilde{k}c_F - \tilde{h}s_F)\,\mathrm{d}F \tag{22}$$

$$\dot{\tilde{k}} = \frac{1}{2\pi}f_t\int_{-\pi}^{\pi}\left(\frac{\partial k}{\partial \dot{\boldsymbol{r}}}\right)^T\cdot\hat{\boldsymbol{u}}(1 - \tilde{k}c_F - \tilde{h}s_F)\,\mathrm{d}F \tag{23}$$

$$\dot{\tilde{p}} = \frac{1}{2\pi}f_t\int_{-\pi}^{\pi}\left(\frac{\partial p}{\partial \dot{\boldsymbol{r}}}\right)^T\cdot\hat{\boldsymbol{u}}(1 - \tilde{k}c_F - \tilde{h}s_F)\,\mathrm{d}F \tag{24}$$

$$\dot{\tilde{q}} = \frac{1}{2\pi}f_t\int_{-\pi}^{\pi}\left(\frac{\partial q}{\partial \dot{\boldsymbol{r}}}\right)^T\cdot\hat{\boldsymbol{u}}(1 - \tilde{k}c_F - \tilde{h}s_F)\,\mathrm{d}F \tag{25}$$

$$\dot{\tilde{\lambda}} = \frac{1}{2\pi}f_t\int_{-\pi}^{\pi}\left(\frac{\partial \lambda}{\partial \dot{\boldsymbol{r}}}\right)^T\cdot\hat{\boldsymbol{u}}(1 - \tilde{k}c_F - \tilde{h}s_F)\,\mathrm{d}F + \frac{1}{2\pi}\int_{-\pi}^{\pi}\tilde{n}(1 - \tilde{k}c_F - \tilde{h}s_F)\,\mathrm{d}F \tag{26}$$

$$\dot{\tilde{\lambda}}_a = \frac{1}{2\pi}f_t\int_{-\pi}^{\pi} -\tilde{\boldsymbol{\lambda}}_z^T\frac{\partial M}{\partial \tilde{a}}\cdot\hat{\boldsymbol{u}}(1 - \tilde{k}c_F - \tilde{h}s_F)\,\mathrm{d}F + \frac{1}{2\pi}\int_{-\pi}^{\pi} -\tilde{\lambda}_\lambda\frac{\partial \tilde{n}}{\partial \tilde{a}}(1 - \tilde{k}c_F - \tilde{h}s_F)\,\mathrm{d}F \tag{27}$$

$$\dot{\tilde{\lambda}}_h = \frac{1}{2\pi}f_t\int_{-\pi}^{\pi} -\tilde{\boldsymbol{\lambda}}_z^T\frac{\partial M}{\partial \tilde{h}}\cdot\hat{\boldsymbol{u}}(1 - \tilde{k}c_F - \tilde{h}s_F)\,\mathrm{d}F \tag{28}$$

$$\dot{\tilde{\lambda}}_k = \frac{1}{2\pi}f_t\int_{-\pi}^{\pi} -\tilde{\boldsymbol{\lambda}}_z^T\frac{\partial M}{\partial \tilde{k}}\cdot\hat{\boldsymbol{u}}(1 - \tilde{k}c_F - \tilde{h}s_F)\,\mathrm{d}F \tag{29}$$

$$\dot{\tilde{\lambda}}_p = \frac{1}{2\pi}f_t\int_{-\pi}^{\pi} -\tilde{\boldsymbol{\lambda}}_z^T\frac{\partial M}{\partial \tilde{p}}\cdot\hat{\boldsymbol{u}}(1 - \tilde{k}c_F - \tilde{h}s_F)\,\mathrm{d}F \tag{30}$$

$$\dot{\tilde{\lambda}}_q = \frac{1}{2\pi}f_t\int_{-\pi}^{\pi} -\tilde{\boldsymbol{\lambda}}_z^T\frac{\partial M}{\partial \tilde{q}}\cdot\hat{\boldsymbol{u}}(1 - \tilde{k}c_F - \tilde{h}s_F)\,\mathrm{d}F \tag{31}$$

$$\dot{\tilde{\lambda}}_\lambda = \frac{1}{2\pi}f_t\int_{-\pi}^{\pi} -\tilde{\boldsymbol{\lambda}}_z^T\frac{\partial M}{\partial \tilde{\lambda}}\cdot\hat{\boldsymbol{u}}(1 - \tilde{k}c_F - \tilde{h}s_F)\,\mathrm{d}F \tag{32}$$

The thrust direction $\hat{\boldsymbol{u}}$ is a function of $\tilde{z}$ and F as well as $\tilde{\boldsymbol{\lambda}}_z$, and the second term in Eq. (26) is equal to zero. An eight-order Gauss–Legendre quadrature is used to evaluate the averaged rates given by Eqs. (21–32), which then are used by a variable step-size integrator to integrate trajectories from given initial conditions, with step sizes spanning several orbital revolutions. The two-point boundary-value problem is identical to the one used in Sec. III, except that all variables as well as the Hamiltonian have averaged values. The averaged Hamiltonian is obtained from the integrated variables

$$\tilde{H} = \tilde{\lambda}_a\dot{\tilde{a}} + \tilde{\lambda}_h\dot{\tilde{h}} + \tilde{\lambda}_k\dot{\tilde{k}} + \tilde{\lambda}_p\dot{\tilde{p}} + \tilde{\lambda}_q\dot{\tilde{q}} + \tilde{\lambda}_\lambda\dot{\tilde{\lambda}}$$

The same example is used with the optimized $(\lambda)_0$ or, equivalently, $(M)_0 = -130.209352$ deg, yielding the solution given by $(\tilde{\lambda}_a)_0 = 0.597196701 \times 10^{-4}$ day/km, $(\tilde{\lambda}_h)_0 = -0.167705951 \times 10^{-11}$ day, $(\tilde{\lambda}_k)_0 = 0.502638000 \times 10^{-3}$ day, $(\tilde{\lambda}_p)_0 = -0.161888685 \times 10^{-12}$ day, $(\tilde{\lambda}_q)_0 = -0.967633609$ day, $(\tilde{\lambda}_\lambda)_0 = 0$ day/rad, and $t_f = 0.6566499459$ days or 56,734.555 s. The achieved final orbit is given by $\tilde{a}_f = 41{,}999.999$ km, $\tilde{e}_f = 9.999999 \times 10^{-4}$, $\tilde{i}_f = 1.000000$ deg, $\tilde{\Omega}_f = 360.000000$ deg, $\tilde{\omega}_f = 359.999999$ deg, and $\tilde{M}_f = 127.049617$ deg with $\tilde{\lambda}_f = 21.066990$ rad, $(\tilde{\lambda}_\lambda)_f = 0$ day/rad, and $\tilde{H}_f = 1.000000007$ indicating a converged solution. Because $(\tilde{\lambda})_0 = -2.272581$ rad, the transfer requires $(\tilde{\lambda}_f - \tilde{\lambda}_0)/(2\pi) = 3.7146$ revolutions of the Earth. This information is available

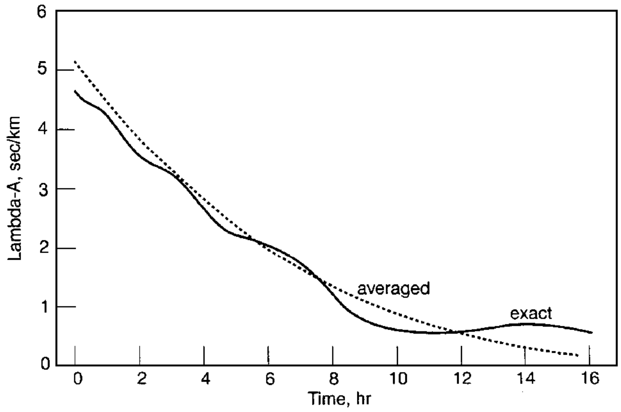

Fig. 7 Comparison between exact and averaged solutions for λ_a for a LEO-to-near-GEO transfer.

because we also have integrated the $\dot{\tilde{\lambda}}$ equation. The multiplier $\tilde{\lambda}_\lambda$ stays constant at zero throughout the transfer as it should, due to the averaging procedure, which eliminates all sensitivity to the position along the orbit. This is verified by starting with $(\tilde{\lambda})_0 = 0$ rad or $\tilde{M}_0 = 0$ deg and generating the exact duplicate trajectory except that $(\tilde{\lambda})_f = 23.339571$ rad or $\tilde{M}_f = 257.258969$ deg such that the transfer still requires 3.7146 revolutions of the Earth. This averaged transfer requires some 1355.476 s less time than the precision-integrated exact transfer of Sec. III, which required $t_f = 58{,}090.031$ s. Therefore, the averaged transfer solution is optimistic by some 22.591 min, requiring a ΔV of 5.559986 km/s, which is close to the ΔV of the exact solution, namely, $\Delta V = 5.692823$ km/s. To compare the various λ, we first must convert the units from days to seconds such that $(\tilde{\lambda}_a)_0 = 5.159779497$ s/km, $(\tilde{\lambda}_h)_0 = -1.448979417 \times 10^{-7}$ s, $(\tilde{\lambda}_k)_0 = 4.342792320 \times 10^1$ s, $(\tilde{\lambda}_p)_0 = -1.398718238 \times 10^{-8}$ s, $(\tilde{\lambda}_q)_0 = -8.360354382 \times 10^4$ s, and $(\tilde{\lambda}_\lambda)_0 = 0$ s/rad. The averaged classical elements $\tilde{\Omega}$ and $\tilde{\omega}$ stay at zero, while the eccentricity $\tilde{e}$ slowly increases from 0 to 10^{-3} at the final time. The $\tilde{\lambda}_a$ multiplier is shown in Fig. 7 with its exact counterpart. The effect of averaging is to smooth out all of the fluctuations seen in the exact curves. The main shortcoming of the averaging of the dynamics is the inability to describe the important eccentricity buildup seen in the exact solution. However, the ΔV evaluation or, equivalently, transfer time is accurate even for this difficult, short duration, large transfer problem. If we now use a very low acceleration f_t of 3.5×10^{-7} km/s^2, this averaged transfer will be achieved in the minimum time of 183.861985 days or $t_f = 1.588567550 \times 10^7$ s. This corresponds to a ΔV of 5.559986 km/s, which is identical to the ΔV required by the fast transfer using averaging. This shows that the averaged transfer time is exactly inversely proportional to the acceleration such that $\Delta V = f_t \cdot t_f$ remains constant, regardless of the duration of the transfer. The exact solution for this case would require very long integration times, and it has not been attempted. However, it is expected that the exact ΔV will be very close to the value of 5.559986 km/s because the transfer orbit will remain nearly circular throughout the exact transfer; this is the case with the averaged solution, which also uses a variable-thrust yaw and pitch profile within each revolution during the ascent. Gravity losses are inherently more important when the orbit departs farther from the circular shape, as is the case with the higher acceleration of $f_t = 9.8 \times 10^{-5}$ km/s^2. The averaged solution fails to account effectively for the gravity losses because it provides an optimistic ΔV with respect to the exact solution for the intermediate acceleration case, namely, 5.559986 vs 5.692823 km/s or a difference of 132.837 m/s. Therefore, we can state that this small ΔV difference also represents the ΔV loss of the exact fast transfer with the intermediate acceleration of $f_t = 9.8 \times 10^{-5}$ km/s^2 with respect to the exact very long duration transfer with the very low acceleration of $f_t = 3.5 \times 10^{-7}$ km/s^2.

V. Conclusion

Exact and approximate optimized transfer solutions for intermediate acceleration applications are presented. Although the algorithms used here are of a general nature, we have restricted ourselves to the important LEO-to-GEO example to characterize the sensitivity of the solution to the orbital position and emphasize the need to consider the full six-state exact dynamics for accurate and reliable results as opposed to the approximate averaged solutions, which are inherently insensitive to orbital position. Furthermore, the orbit parameter histories are poorly simulated with the averaged dynamics, such that their use in actual guidance applications is rather problematic for these fast transfers. More accurate modeling would require the consideration of the effect of the second zonal harmonic J_2 to properly account for the precession of the right ascension of the ascending node, although this effect is small for short duration transfers, such as those described here.

Appendix: Partials of the *M* Matrix

A. Partial Derivatives of *M* with Respect to *h*

$$\frac{\partial M_{11}}{\partial h} = \frac{2}{n^2 a}\frac{\partial \dot{X}_1}{\partial h} \tag{A1}$$

$$\frac{\partial M_{12}}{\partial h} = \frac{2}{n^2 a}\frac{\partial \dot{Y}_1}{\partial h} \tag{A2}$$

$$\frac{\partial M_{13}}{\partial h} = 0 \tag{A3}$$

$$\frac{\partial M_{21}}{\partial h} = \frac{-h}{na^2G}\left(\frac{\partial X_1}{\partial k} - \frac{h\beta}{n}\dot{X}_1\right) + \frac{G}{na^2}\left[\frac{\partial^2 X_1}{\partial h \partial k} - \frac{\dot{X}_1}{n}\left(\beta + \frac{h^2\beta^3}{1-\beta}\right) - \frac{h\beta}{n}\frac{\partial \dot{X}_1}{\partial h}\right] \tag{A4}$$

$$\frac{\partial M_{22}}{\partial h} = \frac{-h}{na^2G}\left(\frac{\partial Y_1}{\partial k} - \frac{h\beta}{n}\dot{Y}_1\right) + \frac{G}{na^2}\left[\frac{\partial^2 Y_1}{\partial h \partial k} - \frac{\dot{Y}_1}{n}\left(\beta + \frac{h^2\beta^3}{1-\beta}\right) - \frac{h\beta}{n}\frac{\partial \dot{Y}_1}{\partial h}\right] \tag{A5}$$

$$\frac{\partial M_{23}}{\partial h} = \frac{hkG^{-3}}{na^2}(qY_1 - pX_1) + \left[k\left(q\frac{\partial Y_1}{\partial h} - p\frac{\partial X_1}{\partial h}\right)\right] \Big/ na^2G \tag{A6}$$

$$\frac{\partial M_{31}}{\partial h} = \frac{h}{na^2G}\left(\frac{\partial X_1}{\partial h} + k\beta\frac{\dot{X}_1}{n}\right) - \frac{G}{na^2}\left(\frac{\partial^2 X_1}{\partial h^2} + \frac{hk\beta^3}{1-\beta}\frac{\dot{X}_1}{n} + \frac{k\beta}{n}\frac{\partial \dot{X}_1}{\partial h}\right) \tag{A7}$$

$$\frac{\partial M_{32}}{\partial h} = \frac{h}{na^2G}\left(\frac{\partial Y_1}{\partial h} + k\beta\frac{\dot{Y}_1}{n}\right) - \frac{G}{na^2}\left(\frac{\partial^2 Y_1}{\partial h^2} + \frac{hk\beta^3}{1-\beta}\frac{\dot{Y}_1}{n} + \frac{k\beta}{n}\frac{\partial \dot{Y}_1}{\partial h}\right) \tag{A8}$$

$$\frac{\partial M_{33}}{\partial h} = \frac{-1}{na^2G}\left[(qY_1 - pX_1) + h\left(q\frac{\partial Y_1}{\partial h} - p\frac{\partial X_1}{\partial h}\right)\right] - \frac{h^2(qY_1 - pX_1)}{na^2G^3} \tag{A9}$$

$$\frac{\partial M_{43}}{\partial h} = \frac{K}{2na^2G}\left[\frac{\partial Y_1}{\partial h} + \frac{hY_1}{G^2}\right] \tag{A10}$$

$$\frac{\partial M_{53}}{\partial h} = \frac{K}{2na^2G}\left[\frac{\partial X_1}{\partial h} + \frac{hX_1}{G^2}\right] \tag{A11}$$

The partials

$$\frac{\partial M_{41}}{\partial h} = \frac{\partial M_{42}}{\partial h} = \frac{\partial M_{51}}{\partial h} = \frac{\partial M_{52}}{\partial h} = 0$$

are all identically zero, and

$$\frac{\partial M_{61}}{\partial h} = \frac{1}{na^2}\left\{-2\frac{\partial X_1}{\partial h} - h\beta G^{-1}\left(h\frac{\partial X_1}{\partial h} + k\frac{\partial X_1}{\partial k}\right) + G\left[\left(\beta + \frac{h^2\beta^3}{1-\beta}\right)\frac{\partial X_1}{\partial h} + \frac{hk\beta^3}{1-\beta}\frac{\partial X_1}{\partial k} + \beta\left(h\frac{\partial^2 X_1}{\partial h^2} + k\frac{\partial^2 X_1}{\partial h\partial k}\right)\right]\right\} \tag{A12}$$

$$\frac{\partial M_{62}}{\partial h} = \frac{1}{na^2}\left\{-2\frac{\partial Y_1}{\partial h} - h\beta G^{-1}\left(h\frac{\partial Y_1}{\partial h} + k\frac{\partial Y_1}{\partial k}\right) + G\left[\left(\beta + \frac{h^2\beta^3}{1-\beta}\right)\frac{\partial Y_1}{\partial h} + \frac{hk\beta^3}{1-\beta}\frac{\partial Y_1}{\partial k} + \beta\left(h\frac{\partial^2 Y_1}{\partial h^2} + k\frac{\partial^2 Y_1}{\partial h\partial k}\right)\right]\right\} \tag{A13}$$

$$\frac{\partial M_{63}}{\partial h} = \frac{G^{-1}}{na^2}\left[\left(q\frac{\partial Y_1}{\partial h} - p\frac{\partial X_1}{\partial h}\right) + hG^{-2}(qY_1 - pX_1)\right] \tag{A14}$$

B. Partial Derivatives of M with Respect to k

$$\frac{\partial M_{11}}{\partial k} = \frac{2}{n^2 a}\frac{\partial \dot{X}_1}{\partial k} \tag{A15}$$

$$\frac{\partial M_{12}}{\partial k} = \frac{2}{n^2 a}\frac{\partial \dot{Y}_1}{\partial k} \tag{A16}$$

$$\frac{\partial M_{13}}{\partial k} = 0 \tag{A17}$$

$$\frac{\partial M_{21}}{\partial k} = \frac{-k}{na^2 G}\left(\frac{\partial X_1}{\partial k} - \frac{h\beta}{n}\dot{X}_1\right) + \frac{G}{na^2}\left[\frac{\partial^2 X_1}{\partial k^2} - \frac{hk\beta^3}{n(1-\beta)}\dot{X}_1 - \frac{h\beta}{n}\frac{\partial \dot{X}_1}{\partial k}\right] \tag{A18}$$

$$\frac{\partial M_{22}}{\partial k} = \frac{-k}{na^2 G}\left(\frac{\partial Y_1}{\partial k} - \frac{h\beta}{n}\dot{Y}_1\right) + \frac{G}{na^2}\left[\frac{\partial^2 Y_1}{\partial k^2} - \frac{hk\beta^3}{n(1-\beta)}\dot{Y}_1 - \frac{h\beta}{n}\frac{\partial \dot{Y}_1}{\partial k}\right] \tag{A19}$$

$$\frac{\partial M_{23}}{\partial k} = \frac{(qY_1 - pX_1)}{na^2 G} + \frac{1}{na^2 G} \times \left[k\left(q\frac{\partial Y_1}{\partial k} - p\frac{\partial X_1}{\partial k}\right) + \frac{k^2(qY_1 - pX_1)}{G^2}\right] \tag{A20}$$

$$\frac{\partial M_{31}}{\partial k} = \frac{k}{na^2 G}\left(\frac{\partial X_1}{\partial h} + k\beta\frac{\dot{X}_1}{n}\right) - \frac{G}{na^2} \times \left[\frac{\partial^2 X_1}{\partial k\partial h} + \left(\beta + \frac{k^2\beta^3}{1-\beta}\right)\frac{\dot{X}_1}{n} + \frac{k\beta}{n}\frac{\partial \dot{X}_1}{\partial k}\right] \tag{A21}$$

$$\frac{\partial M_{32}}{\partial k} = \frac{k}{na^2 G}\left(\frac{\partial Y_1}{\partial h} + k\beta\frac{\dot{Y}_1}{n}\right) - \frac{G}{na^2} \times \left[\frac{\partial^2 Y_1}{\partial k\partial h} + \left(\beta + \frac{k^2\beta^3}{1-\beta}\right)\frac{\dot{Y}_1}{n} + \frac{k\beta}{n}\frac{\partial \dot{Y}_1}{\partial k}\right] \tag{A22}$$

$$\frac{\partial M_{33}}{\partial k} = \frac{-h}{na^2 G}\left(q\frac{\partial Y_1}{\partial k} - p\frac{\partial X_1}{\partial k}\right) - \frac{hk}{na^2 G^3}(qY_1 - pX_1) \tag{A23}$$

$$\frac{\partial M_{43}}{\partial k} = \frac{K}{2na^2 G}\frac{\partial Y_1}{\partial k} + \frac{kK}{2na^2 G^3}Y_1 \tag{A24}$$

$$\frac{\partial M_{53}}{\partial k} = \frac{K}{2na^2 G}\frac{\partial X_1}{\partial k} + \frac{kK}{2na^2 G^3}X_1 \tag{A25}$$

$$\frac{\partial M_{41}}{\partial k} = \frac{\partial M_{42}}{\partial k} = \frac{\partial M_{51}}{\partial k} = \frac{\partial M_{52}}{\partial k} = 0$$

$$\frac{\partial M_{61}}{\partial k} = \frac{1}{na^2}\left\{-2\frac{\partial X_1}{\partial k} - k\beta G^{-1}\left(h\frac{\partial X_1}{\partial h} + k\frac{\partial X_1}{\partial k}\right) + G\left[\left(\beta + \frac{k^2\beta^3}{1-\beta}\right)\frac{\partial X_1}{\partial k} + \frac{hk\beta^3}{1-\beta}\frac{\partial X_1}{\partial h} + \beta\left(h\frac{\partial^2 X_1}{\partial k\partial h} + k\frac{\partial^2 X_1}{\partial k^2}\right)\right]\right\} \tag{A26}$$

$$\frac{\partial M_{62}}{\partial k} = \frac{1}{na^2}\left\{-2\frac{\partial Y_1}{\partial k} - k\beta G^{-1}\left(h\frac{\partial Y_1}{\partial h} + k\frac{\partial Y_1}{\partial k}\right) + G\left[\left(\beta + \frac{k^2\beta^3}{1-\beta}\right)\frac{\partial Y_1}{\partial k} + \frac{hk\beta^3}{1-\beta}\frac{\partial Y_1}{\partial h} + \beta\left(h\frac{\partial^2 Y_1}{\partial k\partial h} + k\frac{\partial^2 Y_1}{\partial k^2}\right)\right]\right\} \tag{A27}$$

$$\frac{\partial M_{63}}{\partial k} = \frac{G^{-1}}{na^2}\left[\left(q\frac{\partial Y_1}{\partial k} - p\frac{\partial X_1}{\partial k}\right) + kG^{-2}(qY_1 - pX_1)\right] \tag{A28}$$

C. Partial Derivatives of M with Respect to p

The nonzero partials are

$$\frac{\partial M_{23}}{\partial p} = \frac{-kX_1}{na^2 G} \tag{A29}$$

$$\frac{\partial M_{33}}{\partial p} = \frac{hX_1}{na^2 G} \tag{A30}$$

$$\frac{\partial M_{43}}{\partial p} = \frac{pY_1}{na^2 G} \tag{A31}$$

$$\frac{\partial M_{53}}{\partial p} = \frac{pX_1}{na^2 G} \tag{A32}$$

$$\frac{\partial M_{63}}{\partial p} = \frac{-X_1}{na^2 G} \tag{A33}$$

D. Partial Derivatives of M with Respect to q

The nonzero partials are

$$\frac{\partial M_{23}}{\partial q} = \frac{kY_1}{na^2 G} \tag{A34}$$

$$\frac{\partial M_{33}}{\partial q} = \frac{-hY_1}{na^2 G} \tag{A35}$$

$$\frac{\partial M_{43}}{\partial q} = \frac{qY_1}{na^2 G} \tag{A36}$$

$$\frac{\partial M_{53}}{\partial q} = \frac{qX_1}{na^2 G} \tag{A37}$$

$$\frac{\partial M_{63}}{\partial q} = \frac{Y_1}{na^2 G} \tag{A38}$$

The partial derivatives of $\dot{X}_1$ with respect to h and k are

$$\frac{\partial \dot{X}_1}{\partial h} = \frac{a}{r}\dot{X}_1\left[s_F + \frac{a}{r}c_F(ks_F - hc_F)\right] + \frac{na^2}{r} \times \left\{h\beta s_F + (kc_F + hs_F)\left(\beta + \frac{h^2\beta^3}{1-\beta}\right) + \frac{a}{r}c_F\left[hk\beta s_F + (1-h^2\beta)c_F\right]\right\} \quad \text{(A39)}$$

$$\frac{\partial \dot{X}_1}{\partial k} = -\frac{\dot{X}_1}{r}a\left[-c_F + \frac{a}{r}s_F(ks_F - hc_F)\right] + \frac{na^2}{r}\left\{\frac{hk\beta^3}{1-\beta}(kc_F + hs_F) + h\beta c_F - \frac{a}{r}s_F\left[hk\beta s_F + (1-h^2\beta)c_F\right]\right\} \quad \text{(A40)}$$

The partials of $\dot{Y}_1$ with respect to h and k are

$$\frac{\partial \dot{Y}_1}{\partial h} = -\frac{\dot{Y}_1}{r}a\left[-s_F - \frac{a}{r}c_F(ks_F - hc_F)\right] + \frac{na^2}{r}\left\{-\frac{hk\beta^3}{1-\beta}(kc_F + hs_F) - k\beta s_F + \left[hk\beta c_F + (1-k^2\beta)s_F\right]\frac{a}{r}c_F\right\} \quad \text{(A41)}$$

$$\frac{\partial \dot{Y}_1}{\partial k} = \frac{-\dot{Y}_1}{r}a\left[-c_F + \frac{a}{r}s_F(ks_F - hc_F)\right] + \frac{na^2}{r} \times \left\{-\left(\beta + \frac{k^2\beta^3}{1-\beta}\right)(kc_F + hs_F) - k\beta c_F - \frac{a}{r}s_F\left[hk\beta c_F + (1-k^2\beta)s_F\right]\right\} \quad \text{(A42)}$$

The second partials of X_1 and Y_1 with respect to h and k are

$$\frac{\partial^2 X_1}{\partial h^2} = a\left\{-\frac{2a}{r}c_F\left(\beta + \frac{h^2\beta^3}{1-\beta}\right) - \frac{h\beta^3}{1-\beta}(hc_F - ks_F)\left[3 + \frac{h^2\beta^2(3-2\beta)}{(1-\beta)^2}\right] + \frac{a^2}{r^2}c_F(h\beta - s_F)\left[-s_F + \frac{a}{r}(h - s_F)\right] - \frac{a^2}{r^2}c_F^3\right\} \quad \text{(A43)}$$

$$\frac{\partial^2 X_1}{\partial k^2} = -a\left\{-\frac{2a}{r}s_F\frac{hk\beta^3}{1-\beta} + (hc_F - ks_F)\left[1 + \frac{k^2\beta^2(3-2\beta)}{(1-\beta)^2}\right]\frac{h\beta^3}{1-\beta} + \frac{a^2}{r^2}s_F(h\beta - s_F)\left[-c_F + \frac{a}{r}(k - c_F)\right] + \frac{a^2}{r^2}c_Fs_F^2\right\} \quad \text{(A44)}$$

$$\frac{\partial^2 X_1}{\partial h \partial k} = -a\left\{\frac{a}{r}c_F\frac{hk\beta^3}{1-\beta} + (hc_F - ks_F)\left[1 + \frac{h^2\beta^2(3-2\beta)}{(1-\beta)^2}\right] \times \frac{k\beta^3}{1-\beta} + (s_F - h\beta)\left[\frac{a}{r}\left(s_F^2 - hs_F\right) - c_F^2\right]\frac{a^2}{r^2} - \frac{a^2}{r^2}s_Fc_F^2 - \frac{a}{r}s_F\left(\beta + \frac{h^2\beta^3}{1-\beta}\right)\right\} \quad \text{(A45)}$$

$$\frac{\partial^2 X_1}{\partial k \partial h} = a\left\{\frac{a}{r}s_F\left(\beta + \frac{h^2\beta^3}{1-\beta}\right) - (hc_F - ks_F) \times \left[1 + \frac{h^2\beta^2(3-2\beta)}{(1-\beta)^2}\right]\frac{k\beta^3}{1-\beta} + \frac{a^2}{r^2}\left[\frac{a}{r}\left(kc_F - c_F^2\right) + s_F^2\right] \times (h\beta - s_F) - \frac{a}{r}c_F\frac{hk\beta^3}{1-\beta} + \frac{a^2}{r^2}c_F^2s_F\right\} \quad \text{(A46)}$$

$$\frac{\partial^2 Y_1}{\partial h^2} = a\left\{\frac{2a}{r}c_F\frac{hk\beta^3}{1-\beta} + (hc_F - ks_F)\frac{k\beta^3}{1-\beta}\left[1 + \frac{h^2\beta^2(3-2\beta)}{(1-\beta)^2}\right] + \frac{a^2}{r^2}c_F\left[-\frac{a}{r}(h - s_F) + s_F\right](k\beta - c_F) - \frac{a^2}{r^2}s_Fc_F^2\right\} \quad \text{(A47)}$$

$$\frac{\partial^2 Y_1}{\partial k^2} = a\left\{\frac{-2a}{r}s_F\left(\beta + \frac{k^2\beta^3}{1-\beta}\right) + (hc_F - ks_F)\left[3 + \frac{k^2\beta^2(3-2\beta)}{(1-\beta)^2}\right]\frac{k\beta^3}{1-\beta} + \frac{a^2}{r^2}s_F\left[-\frac{a}{r}(k - c_F) + c_F\right](c_F - k\beta) - \frac{a^2}{r^2}s_F^3\right\} \quad \text{(A48)}$$

$$\frac{\partial^2 Y_1}{\partial h \partial k} = a\left\{\frac{a}{r}c_F\left(\beta + \frac{k^2\beta^3}{1-\beta}\right) + (hc_F - ks_F)\frac{h\beta^3}{1-\beta} \times \left[1 + \frac{k^2\beta^2(3-2\beta)}{(1-\beta)^2}\right] - \frac{a^2}{r^2}\left[\frac{a}{r}s_F(h - s_F) + c_F^2\right] \times (c_F - k\beta) + \frac{a^2}{r^2}c_Fs_F^2 - \frac{a}{r}s_F\frac{hk\beta^3}{1-\beta}\right\} \quad \text{(A49)}$$

$$\frac{\partial^2 Y_1}{\partial k \partial h} = a\left\{-\frac{a}{r}s_F\frac{hk\beta^3}{1-\beta} + (hc_F - ks_F)\frac{h\beta^3}{1-\beta} \times \left[1 + \frac{k^2\beta^2(3-2\beta)}{(1-\beta)^2}\right] - \frac{a^2}{r^2}\left[\frac{a}{r}c_F(k - c_F) + s_F^2\right] \times (k\beta - c_F) + \frac{a}{r}c_F\left(\beta + \frac{k^2\beta^3}{1-\beta}\right) + \frac{a^2}{r^2}c_Fs_F^2\right\} \quad \text{(A50)}$$

The accessory partials $\partial^2 X_1/\partial a \partial k$, $\partial^2 X_1/\partial a \partial h$, $\partial^2 Y_1/\partial a \partial k$, and $\partial^2 Y_1/\partial a \partial h$ are generated with $\partial F/\partial a = 0$. In all of the following partials, $\partial X_1/\partial a = X_1/a$ and $\partial Y_1/\partial a = Y_1/a$:

$$\frac{\partial^2 X_1}{\partial a \partial k} = \frac{1}{a}\frac{\partial X_1}{\partial k} \quad \text{(A51)}$$

$$\frac{\partial^2 X_1}{\partial a \partial h} = \frac{1}{a}\frac{\partial X_1}{\partial h} \quad \text{(A52)}$$

$$\frac{\partial^2 Y_1}{\partial a \partial k} = \frac{1}{a}\frac{\partial Y_1}{\partial k} \quad \text{(A53)}$$

$$\frac{\partial^2 Y_1}{\partial a \partial h} = \frac{1}{a}\frac{\partial Y_1}{\partial h} \quad \text{(A54)}$$

E. Partial Derivatives of *M* with Respect to *a*

$$\frac{\partial M_{11}}{\partial a} = \frac{4}{n^2a^2}\dot{X}_1 + \frac{2}{n^2a}\frac{\partial \dot{X}_1}{\partial a} \quad \text{(A55)}$$

$$\frac{\partial M_{12}}{\partial a} = \frac{4}{n^2a^2}\dot{Y}_1 + \frac{2}{n^2a}\frac{\partial \dot{Y}_1}{\partial a} \tag{A56}$$

$$\frac{\partial M_{13}}{\partial a} = 0 \tag{A57}$$

$$\frac{\partial M_{21}}{\partial a} = \frac{G}{na^2}\left[-\frac{1}{2a}\frac{\partial X_1}{\partial k} + \frac{\partial^2 X_1}{\partial a \partial k} - \frac{h\beta}{na}\dot{X}_1 - \frac{h\beta}{n}\frac{\partial \dot{X}_1}{\partial a}\right] \tag{A58}$$

$$\frac{\partial M_{22}}{\partial a} = \frac{G}{na^2}\left[-\frac{1}{2a}\frac{\partial Y_1}{\partial k} + \frac{\partial^2 Y_1}{\partial a \partial k} - \frac{h\beta}{na}\dot{Y}_1 - \frac{h\beta}{n}\frac{\partial \dot{Y}_1}{\partial a}\right] \tag{A59}$$

$$\frac{\partial M_{23}}{\partial a} = \frac{k}{na^2G}\left[-\frac{1}{2a}(qY_1 - pX_1) + q\frac{\partial Y_1}{\partial a} - p\frac{\partial X_1}{\partial a}\right] \tag{A60}$$

$$\frac{\partial M_{31}}{\partial a} = -\frac{G}{na^2}\left[-\frac{1}{2a}\frac{\partial X_1}{\partial h} + \frac{\partial^2 X_1}{\partial a \partial h} + \frac{k\beta}{na}\dot{X}_1 + \frac{k\beta}{n}\frac{\partial \dot{X}_1}{\partial a}\right] \tag{A61}$$

$$\frac{\partial M_{32}}{\partial a} = -\frac{G}{na^2}\left[-\frac{1}{2a}\frac{\partial Y_1}{\partial h} + \frac{\partial^2 Y_1}{\partial a \partial h} + \frac{k\beta}{na}\dot{Y}_1 + \frac{k\beta}{n}\frac{\partial \dot{Y}_1}{\partial a}\right] \tag{A62}$$

$$\frac{\partial M_{33}}{\partial a} = \frac{-h}{na^2G}\left[-\frac{1}{2a}(qY_1 - pX_1) + q\frac{\partial Y_1}{\partial a} - p\frac{\partial X_1}{\partial a}\right] \tag{A63}$$

$$\frac{\partial M_{41}}{\partial a} = 0 \tag{A64}$$

$$\frac{\partial M_{42}}{\partial a} = 0 \tag{A65}$$

$$\frac{\partial M_{43}}{\partial a} = \frac{K}{2na^2G}\left(-\frac{1}{2a}Y_1 + \frac{\partial Y_1}{\partial a}\right) \tag{A66}$$

$$\frac{\partial M_{51}}{\partial a} = 0 \tag{A67}$$

$$\frac{\partial M_{52}}{\partial a} = 0 \tag{A68}$$

$$\frac{\partial M_{53}}{\partial a} = \frac{K}{2na^2G}\left(-\frac{1}{2a}X_1 + \frac{\partial X_1}{\partial a}\right) \tag{A69}$$

$$\frac{\partial M_{61}}{\partial a} = \frac{-M_{61}}{2a} + \frac{1}{na^2}\left[-2\frac{\partial X_1}{\partial a} + G\left(h\beta\frac{\partial^2 X_1}{\partial a \partial h} + k\beta\frac{\partial^2 X_1}{\partial a \partial k}\right)\right] \tag{A70}$$

$$\frac{\partial M_{62}}{\partial a} = -\frac{M_{62}}{2a} + \frac{1}{na^2}\left[-2\frac{\partial Y_1}{\partial a} + G\left(h\beta\frac{\partial^2 Y_1}{\partial a \partial h} + k\beta\frac{\partial^2 Y_1}{\partial a \partial k}\right)\right] \tag{A71}$$

$$\frac{\partial M_{63}}{\partial a} = -\frac{M_{63}}{2a} + \frac{1}{na^2}\left[\left(q\frac{\partial Y_1}{\partial a} - p\frac{\partial X_1}{\partial a}\right)G^{-1}\right] \tag{A72}$$

with

$$\frac{\partial \dot{X}_1}{\partial a} = -\frac{1}{2}\frac{na}{r}\left[hk\beta c_F - (1 - h^2\beta)s_F\right] \tag{A73}$$

$$\frac{\partial \dot{Y}_1}{\partial a} = \frac{1}{2}\frac{na}{r}\left[hk\beta s_F - (1 - k^2\beta)c_F\right] \tag{A74}$$

F. Partial Derivatives of M with Respect to λ

$$\frac{\partial M_{11}}{\partial \lambda} = \frac{2}{n^2r}\frac{\partial \dot{X}_1}{\partial F} \tag{A75}$$

$$\frac{\partial M_{12}}{\partial \lambda} = \frac{2}{n^2r}\frac{\partial \dot{Y}_1}{\partial F} \tag{A76}$$

$$\frac{\partial M_{13}}{\partial \lambda} = 0 \tag{A77}$$

$$\frac{\partial M_{21}}{\partial \lambda} = \frac{G}{nar}\left(\frac{\partial^2 X_1}{\partial F \partial k} - \frac{h\beta}{n}\frac{\partial \dot{X}_1}{\partial F}\right) \tag{A78}$$

$$\frac{\partial M_{22}}{\partial \lambda} = \frac{G}{nar}\left(\frac{\partial^2 Y_1}{\partial F \partial k} - \frac{h\beta}{n}\frac{\partial \dot{Y}_1}{\partial F}\right) \tag{A79}$$

$$\frac{\partial M_{23}}{\partial \lambda} = \left[k\left(q\frac{\partial Y_1}{\partial F} - p\frac{\partial X_1}{\partial F}\right)\right]\Big/ narG \tag{A80}$$

$$\frac{\partial M_{31}}{\partial \lambda} = -\frac{G}{nar}\left(\frac{\partial^2 X_1}{\partial F \partial h} + \frac{k\beta}{n}\frac{\partial \dot{X}_1}{\partial F}\right) \tag{A81}$$

$$\frac{\partial M_{32}}{\partial \lambda} = -\frac{G}{nar}\left(\frac{\partial^2 Y_1}{\partial F \partial h} + \frac{k\beta}{n}\frac{\partial \dot{Y}_1}{\partial F}\right) \tag{A82}$$

$$\frac{\partial M_{33}}{\partial \lambda} = \left[-h\left(q\frac{\partial Y_1}{\partial F} - p\frac{\partial X_1}{\partial F}\right)\right]\Big/ narG \tag{A83}$$

$$\frac{\partial M_{41}}{\partial \lambda} = \frac{\partial M_{42}}{\partial \lambda} = 0 \tag{A84}$$

$$\frac{\partial M_{43}}{\partial \lambda} = \frac{K}{2narG}\frac{\partial Y_1}{\partial F} \tag{A85}$$

$$\frac{\partial M_{51}}{\partial \lambda} = \frac{\partial M_{52}}{\partial \lambda} = 0 \tag{A86}$$

$$\frac{\partial M_{53}}{\partial \lambda} = \frac{K}{2narG}\frac{\partial X_1}{\partial F} \tag{A87}$$

$$\frac{\partial M_{61}}{\partial \lambda} = \frac{1}{nar}\left[-2\frac{\partial X_1}{\partial F} + G\left(h\beta\frac{\partial^2 X_1}{\partial F \partial h} + k\beta\frac{\partial^2 X_1}{\partial F \partial k}\right)\right] \tag{A88}$$

$$\frac{\partial M_{62}}{\partial \lambda} = \frac{1}{nar}\left[-2\frac{\partial Y_1}{\partial F} + G\left(h\beta\frac{\partial^2 Y_1}{\partial F \partial h} + k\beta\frac{\partial^2 Y_1}{\partial F \partial k}\right)\right] \tag{A89}$$

$$\frac{\partial M_{63}}{\partial \lambda} = \left(q\frac{\partial Y_1}{\partial F} - p\frac{\partial X_1}{\partial F}\right)\Big/ narG \tag{A90}$$

The auxiliary partials are

$$\frac{\partial X_1}{\partial F} = a\left[hk\beta c_F - (1 - h^2\beta)s_F\right] \tag{A91}$$

$$\frac{\partial Y_1}{\partial F} = a\left[-hk\beta s_F + (1 - k^2\beta)c_F\right] \tag{A92}$$

$$\frac{\partial \dot{X}_1}{\partial F} = -\frac{a}{r}(ks_F - hc_F)\dot{X}_1 + \frac{a^2n}{r}\left[-hk\beta s_F - (1 - h^2\beta)c_F\right] \tag{A93}$$

$$\frac{\partial \dot{Y}_1}{\partial F} = -\frac{a}{r}(ks_F - hc_F)\dot{Y}_1 + \frac{a^2n}{r}\left[-hk\beta c_F - (1 - k^2\beta)s_F\right] \tag{A94}$$

$$\frac{\partial^2 X_1}{\partial F \partial h} = a\left[(hs_F + kc_F)\left(\beta + \frac{h^2\beta^3}{1 - \beta}\right) + \frac{a^2}{r^2}(h\beta - s_F)(s_F - h) + \frac{a}{r}c_F^2\right] \tag{A95}$$

$$\frac{\partial^2 X_1}{\partial F \partial k} = -a\left[-(hs_F + kc_F)\frac{hk\beta^3}{1 - \beta} + \frac{a^2}{r^2}(s_F - h\beta)(c_F - h) + \frac{a}{r}s_Fc_F\right] \tag{A96}$$

$$\frac{\partial^2 Y_1}{\partial F \partial h} = a\left[-(hs_F + kc_F)\frac{hk\beta^3}{1-\beta}\right.$$

$$\left. - \frac{a^2}{r^2}(k\beta - c_F)(s_F - h) + \frac{a}{r}s_F c_F\right] \tag{A97}$$

$$\frac{\partial^2 Y_1}{\partial F \partial k} = a\left[-(hs_F + kc_F)\left(\beta + \frac{k^2\beta^3}{1-\beta}\right)\right.$$

$$\left. + \frac{a^2}{r^2}(c_F - k\beta)(c_F - k) - \frac{a}{r}s_F^2\right] \tag{A98}$$

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